Chapter 4. Linear Systems

4.1 Solve Linear Systems Algebraically

A **linear system of equations** consists of two equations in two variables. A solution for a linear system is a set of one value for each variable that solves both equations at the same time. If the variables are x and y, the solution is an ordered pair which represents the point where the two equations' lines intersect on a coordinate plane.

There are two common methods used to solve linear systems algebraically: the addition method and the substitution method. In both methods, we aim to develop a new equation in which one of the two variables has been eliminated.

The **addition method (elimination method)** depends on the fact that if two equations are true, both the left and right sides of each equation can be added to create a new equation:

If $a = b$ and $c = d$, then $a + c = b + d$ or, written vertically →

$$\begin{array}{r} a=b \\ c=d \\ \hline a+c=b+d \end{array}$$

It also depends on the fact that we can multiply both sides of an equation by the same value and the equation remains true:

If $a = b$, then $ma = mb$

The goal in the addition method is to eliminate a variable by adding terms whose coefficients for that variable are additive inverses.

Example:

$$\begin{array}{r} 2x - y = 2 \\ x + y = 4 \\ \hline 3x \quad\;\; = 6 \end{array}$$

Adding $-y$ and $+y$ eliminates the variable y, allowing us to solve a simpler equation in one variable, $3x = 6$, or $x = 2$.

In the addition method, if two variable terms (in the same variable) are not already additive inverses, they can be made into additive inverses by multiplying one or both equations by values that will change the coefficients into inverses.

Example:

$$\begin{array}{rcr} 5a + b = 13 & \times 3 & 15a + 3b = 39 \\ 4a - 3b = 18 & \rightarrow & 4a - 3b = 18 \\ \cline{3-3} & & 19a \quad\;\; = 57 \end{array}$$

$+3b$ and $-3b$ are inverses

Therefore, $a = \frac{57}{19} = 3$.

Once we know the value of one variable, we can substitute that value for the variable in either of the two original equations to solve for the other variable.

Example: In the above example, since we know $a = 3$, we can substitute 3 for a in either of the original equations to find b.

$5(3) + b = 13$

$15 + b = 13$

$b = -2$

In the **substitution method**, an equation needs to have one of the variables expressed in terms of the other, or we will need to solve for one of the variables. Once we have a variable equal to an expression, we can substitute that expression for that variable in the other equation.

Example: $y = x + 1$
$x + 2y = 17 \rightarrow \quad x + 2(x + 1) = 17$
Now, solving the new equation for x gives us
$x + 2(x + 1) = 17$
$x + 2x + 2 = 17$
$3x + 2 = 17$
$3x = 15$
$x = 5$

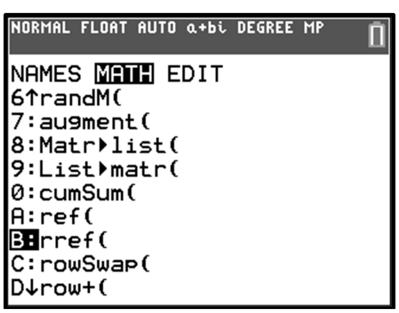
As with the addition method, once we know the value of one variable, we can substitute the known value into either original equation to solve for the other variable.
Example: For the above example, $y = x + 1$, so $y = 5 + 1 = 6$.

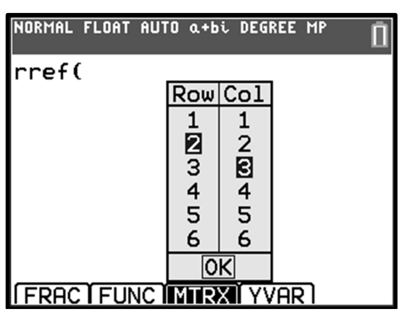
CALCULATOR TIP

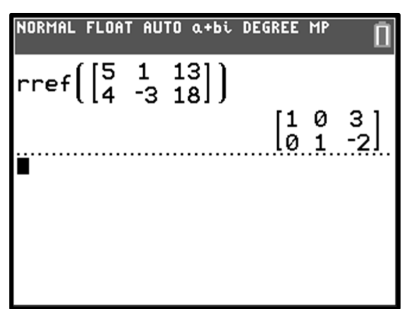
To check algebraic solutions to linear systems using the matrix feature:

1. Press [2nd][MATRIX]<MATH>[ALPHA][B] to select the **rref** function (which stands for "reduced row echelon form").
2. Create a matrix with a size of 2 rows and 3 columns.
   - On the TI-84, press [ALPHA][F3], select 2 and 3 with the arrow keys, then select OK.
   - On the TI-83, press [2nd][MATRIX][1] and type 2×3 for 2 rows and 3 columns.
3. Each row of the matrix represents an equation. Enter the coefficients of the first variable in column 1, coefficients of the second variable in column 2, and constants in column 3.
   - On the TI-84, exit the matrix by pressing [▶]. Then, press [)][ENTER].
   - On the TI-83, press [2nd][QUIT][2nd][MATRIX][1][)][ENTER].
4. The resulting matrix will show the values of the variables, in order, in column 3.

Example: The screenshots below show how the calculator solves the system,
$5a + b = 13$
$4a - 3b = 18$
The solutions matrix shows that $a = 3$ and $b = -2$.







MODEL PROBLEM 1: *ADDITION METHOD*

Solve the following system of equations using the addition method:

$$4a + 2b = 22$$
$$-4a + 3b = 3$$

Solution:

$$4a + 2b = 22$$
$$\underline{-4a + 3b = 3}$$

(A) $5b = 25$

(B) $b = 5$

(C) $4a + 2(5) = 22$

(D) $4a + 10 = 22$

$4a = 12$

$a = 3$

(E) Solution: *a = 3, b = 5*

Explanation of steps:

(A) If the coefficients for one of the variables are additive inverses *[4a and −4a]*, add the equations to derive a new equation without that variable.

(B) Now that we have an equation with only one variable *[b]*, solve for that one variable.

(C) Substitute this solution *[b=5]* into either one of the original equations.

(D) Solve for the other variable [a].

(E) Write the solution by stating the values of both variables.

PRACTICE PROBLEMS

1. Solve the following system of equations for *x* and *y* algebraically and check your solutions on the calculator: $3x - y = 8$ $x + y = 4$	2. Solve the following system of equations for *x* and *y* algebraically and check your solutions on the calculator: $2x - 3y = 19$ $3x + 3y = 21$
3. Solve the following system of equations for *x* and *y* algebraically and check your solutions on the calculator: $3x + 2y = 12$ $5x - 2y = 4$	4. Solve the following system of equations for *x* and *y* algebraically and check your solutions on the calculator: $2x - 5y = 11$ $-2x + 3y = -9$

5. Solve the following system of equations for *x* and *y* algebraically and check your solutions on the calculator: $2x - 4y = 12$ $-2x + y = -9$	6. Solve the following system of equations for *x* and *y* algebraically and check your solutions on the calculator: $3x + y = 0$ $-x - y = -4$

Model Problem 2: *Addition Method with Multipliers*

Solve the following system of equations using the addition method:

$5x + 8y = 1$
$3x + 4y = -1$

Solution:

	(A)	(B)	(C)
$5x + 8y = 1$	→	$5x + 8y = 1$	$5(-3) + 8y = 1$
$3x + 4y = -1$	$\times (-2)$	$-6x - 8y = 2$	$-15 + 8y = 1$
		$-x \quad = 3$	$8y = 16$
		$x = -3$	$y = 2$

(D) Solution: $(-3,2)$

Explanation of steps:

(A) If neither the *x* terms nor the *y* terms are additive inverses of each other, multiply one (or each) of the equations by a value to turn them into inverses *[to change the "y" terms into inverses, 8y and –8y, we can multiply the second equation by –2]*.

(B) Adding the equations eliminates the inverses and will now give us a new equation in one variable. Solve for that variable.

(C) Then, substitute the solution *[*$x = -3$*]* into either one of the original equations *[the first equation was used here]*, allowing you to solve for the other variable *[*$y = 2$*]*.

(D) The solution for variables *x* and *y* may be written as an ordered pair, (*x,y*).

Practice Problems

7. Solve the following system of equations for x and y algebraically and check your solutions on the calculator: $3x + 2y = 4$ $-2x + 2y = 24$	8. What is the value of y in the following system of equations? $2x + 3y = 6$ $2x + y = -2$
9. Solve the following system of equations for x and y algebraically and check your solutions on the calculator: $-3x + 4y = 11$ $6x - 5y = -16$	10. Solve the following system of equations for x and y algebraically and check your solutions on the calculator: $2x + 3y = 7$ $x + y = 3$

11. Solve the following system of equations for x and y algebraically and check your solutions on the calculator:

$$2x + y = 8$$
$$x - 3y = -3$$

12. Solve the following system of equations for x and y algebraically and check your solutions on the calculator:

$$x + 2y = 9$$
$$x - y = 3$$

13. Solve the following system of equations for x and y algebraically and check your solutions on the calculator:

$$3x + 2y = 4$$
$$4x + 3y = 7$$

14. Solve the following system of equations for x and y algebraically and check your solutions on the calculator:

$$3x + 4y = 9$$
$$5x + 6y = 21$$

MODEL PROBLEM 3: *SUBSTITUTION METHOD*

Solve the following system of equations using the substitution method:

$3x - y = 16$
$y = x - 8$

Solution:

(A) $3x - (x - 8) = 16$
(B) $3x - x + 8 = 16$
$2x + 8 = 16$
$2x = 8$
$x = 4$

(C) $y = (4) - 8$
$y = -4$

(D) Solution: $(4, -4)$

Explanation of steps:

(A) If one equation already has one of the variables isolated, as in *variable = expression*, substitute this expression for this variable in the other equation. *[*$y = x - 8$ *already has y expressed in terms of x, so substitute the expression* $x - 8$ *for the y in the first equation:* $3x - \underline{y} = 16$ *becomes* $3x - \underline{(x - 8)} = 16$*.]* It is always safest to use parentheses around the expression whenever you perform a substitution.
(B) Solve the equation for one variable.
(C) Substitute the solution found in step (B) into either original equation *[substitute 4 for x]*.
(D) Solve the equation for the other variable.
(E) State the solution.

PRACTICE PROBLEMS

15. Solve the following system of equations for *x* and *y*: $y = 4x - 10$ $y = 5 - x$	16. Solve the following system of equations for *x* and *y*: $x = y - 2$ $y = 10 - 3x$

17. Solve the following system of equations for x and y:

$$y = 9 - 2x$$
$$3y - 2x = 11$$

18. Using the substitution method, solve the following system of equations for x and y:

$$7x + 3y = 68$$
$$x - 4y = -8$$

19. Solve the following system of equations algebraically:

$$2a + 3b = 12$$
$$a = \frac{1}{2}b - 6$$

20. Solve the following system of equations algebraically:

$$c + 3d = 8$$
$$c = 4d - 6$$

21. To solve the following system of equations by the substitution method, which equivalent equation could be used?

$$2x - y = 5$$
$$3x + 2y = -3$$

(1) $3x + 2(2x - 5) = -3$

(2) $3x + 2(5 - 2x) = -3$

(3) $3\left(y + \frac{5}{2}\right) + 2y = -3$

(4) $3\left(\frac{5}{2} - y\right) + 2y = -3$

Regents Questions

Multiple Choice

1. Which system of equations does *not* have the same solution as the system below?

$$4x + 3y = 10$$
$$-6x - 5y = -16$$

(1) $-12x - 9y = -30$
$12x + 10y = 32$

(2) $20x + 15y = 50$
$-18x - 15y = -48$

(3) $24x + 18y = 60$
$-24x - 20y = -64$

(4) $40x + 30y = 100$
$36x + 30y = -96$

2. A system of equations is shown below.

Equation A: $5x + 9y = 12$
Equation B: $4x - 3y = 8$

Which method eliminates one of the variables?

(1) Multiply equation A by $-\frac{1}{3}$ and add the result to equation B.
(2) Multiply equation B by 3 and add the result to equation A.
(3) Multiply equation A by 2 and equation B by -6 and add the results together.
(4) Multiply equation B by 5 and equation A by 4 and add the results together.

3. Which system of equations will yield the same solution as the system below?

$$x - y = 3$$
$$2x - 3y = -1$$

(1) $-2x - 2y = -6$
$2x - 3y = -1$

(2) $-2x + 2y = 3$
$2x - 3y = -1$

(3) $2x - 2y = 6$
$2x - 3y = -1$

(4) $3x + 3y = 9$
$2x - 3y = -1$

4. Using the substitution method, Vito is solving the following system of equations algebraically:

$$y + 3x = -4$$
$$2x - 3y = -21$$

Which equivalent equation could Vito use?

(1) $2(-3x - 4) + 3x = -21$
(2) $2(3x - 4) + 3x = -21$
(3) $2x - 3(-3x - 4) = -21$
(4) $2x - 3(3x - 4) = -21$

5. Which system of linear equations has the same solution as the one shown below?

$$x - 4y = -10$$
$$x + y = 5$$

(1) $5x = 10$
$x + y = 5$

(2) $-5y = -5$
$x + y = 5$

(3) $-3x = -30$
$x + y = 5$

(4) $-5y = -5$
$x - 4y = -10$

6. Which system of equations has the same solutions as the one shown below?

$$3x - y = 7$$
$$2x + 3y = 12$$

(1) $6x - 2y = 14$
$-6x + 9y = 36$

(2) $18x - 6y = 42$
$4x + 6y = 24$

(3) $-9x - 3y = -21$
$2x + 3y = 12$

(4) $3x - y = 7$
$x + y = 2$

7. Which system has the same solution as the system below?

$$x + 3y = 10$$
$$-2x - 2y = 4$$

(1) $-x + y = 6$
$2x + 6y = 20$

(2) $-x + y = 14$
$2x + 6y = 20$

(3) $x + y = 6$
$2x + 6y = 20$

(4) $x + y = 14$
$2x + 6y = 20$